

Deformed quantum mechanics and q -Hermitian operators

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Abstract. Starting on the basis of the non-commutative q -differential calculus, we introduce a generalized q -deformed Schrödinger equation. It can be viewed as the quantum stochastic counterpart of a generalized classical kinetic equation, which reproduces at the equilibrium the well-known q -deformed exponential stationary distribution. In this framework, q -deformed adjoint of an operator and q -hermitian operator properties occur in a natural way in order to satisfy the basic quantum mechanics assumptions.

1. Introduction

In the recent past, there has been a great deal of interest in the study of quantum algebra and quantum groups in connection between several physical fields [1]. From the seminal work of Biedenharn [2] and Macfarlane [3], it was clear that the q -calculus, originally introduced in the study of the basic hypergeometric series [4, 5, 6], plays a central role in the representation of the quantum groups with a deep physical meaning and not merely a mathematical exercise. Many physical applications have been investigated on the basis of the q -deformation of the Heisenberg algebra [7, 8, 9, 10, 11]. In Ref.[12, 13] it was shown that a natural realization of quantum thermostatics of q -deformed bosons and fermions can be built on the formalism of q -calculus. In Ref.[14], a q -deformed Poisson bracket, invariant under the action of the q -symplectic group, has been derived and a classical q -deformed thermostatics has been proposed in Ref.[15]. Furthermore, it is remarkable to observe that q -calculus is very well suited for to describe fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson q -derivative and q -integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [16].

In the past, the study of generalized linear and non-linear Schrödinger equations has attracted a lot of interest because of many collective effects in quantum many-body models can be described by means of effective theories with generalized one-particle Schrödinger equation [17, 18, 19, 20]. On the other hand, it is relevant to mention that

in the last years many investigations in literature has been devoted to non-Hermitian and pseudo-Hermitian quantum mechanics [21, 22, 23, 24, 25, 27].

In the framework of the q -Heisenberg algebra, a q -deformed Schrödinger equations have been proposed [28, 29]. Although the proposed quantum dynamics is based on the noncommutative differential structure on configuration space, we believe that a fully consistent q -deformed formalism of the quantum dynamics, based on the properties of the q -calculus, has been still lacking.

In this paper, starting on a generalized classical kinetic equation reproducing as stationary distribution of the well-know q -exponential function, we study a generalization of the quantum dynamics consistently with the prescriptions of the q -differential calculus. At this scope, we introduce a q -deformed Schrödinger equation with a deformed Hamiltonian which is a non-Hermitian operator with respect to the standard (undeformed) operators properties but its dynamics satisfies the basic assumptions of the quantum mechanics under generalized operators properties, such as the definition of q -adjoint and q -hermitian operator.

2. Noncommutative differential calculus

We shall briefly review the main features of the noncommutative differential q -calculus for real numbers. It is based on the following q -commutative relation among the operators \hat{x} and $\hat{\partial}_x$,

$$\hat{\partial}_x \hat{x} = 1 + q \hat{x} \hat{\partial}_x , \quad (1)$$

with q a real and positive parameter.

A realization of the above algebra in terms of ordinary real numbers can be accomplished by the replacement [14, 30]

$$\hat{x} \rightarrow x , \quad (2)$$

$$\hat{\partial}_x \rightarrow \mathcal{D}_x^{(q)} , \quad (3)$$

where $\mathcal{D}_x^{(q)}$ is the Jackson derivative [4] defined as

$$\mathcal{D}_x^{(q)} = \frac{D_x^{(q)} - 1}{(q - 1)x} , \quad (4)$$

where

$$D_x^{(q)} = q^{x \partial_x} \quad (5)$$

is the dilatation operator. Its action on an arbitrary real function $f(x)$ is given by

$$\mathcal{D}_x^{(q)} f(x) = \frac{f(qx) - f(x)}{(q - 1)x} . \quad (6)$$

The Jackson derivative satisfies some simple proprieties which will be useful in the following. For instance, its action on a monomial $f(x) = x^n$ is given by

$$\mathcal{D}_x^{(q)} x^n = [n]_q x^{n-1} , \quad (7)$$

and

$$\mathcal{D}_x^{(q)} x^{-n} = -\frac{[n]_q}{q^n} \frac{1}{x^{n+1}} , \quad (8)$$

where $n \geq 0$ and

$$[n]_q = \frac{q^n - 1}{q - 1} , \quad (9)$$

are the so-called *basic*-numbers. Moreover, we can easily verify the following q -version of the Leibnitz rule

$$\begin{aligned} \mathcal{D}_x^{(q)} (f(x) g(x)) &= \mathcal{D}_x^{(q)} f(x) g(x) + f(qx) \mathcal{D}_x^{(q)} g(x) , \\ &= \mathcal{D}_x^{(q)} f(x) g(qx) + f(x) \mathcal{D}_x^{(q)} g(x) . \end{aligned} \quad (10)$$

A relevant role in the q -algebra, as developed by Jackson, is given by the *basic*-binomial series defined by

$$\begin{aligned} (x + y)^{(n)} &= (x + y) (x + qy) (x + q^2 y) \dots (x + q^{n-1} y) \\ &\equiv \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-1)/2} x^{n-r} y^r , \end{aligned} \quad (11)$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!} , \quad (12)$$

is known as the q -binomial coefficient which reduces to the ordinary binomial coefficient in the $q \rightarrow 1$ limit [6]. We should remark that Eq.(12) holds for $0 \leq r \leq n$, while it is assumed to vanish otherwise and we have defined $[n]_q! = [n]_q [n-1]_q \dots [1]_q$. Remarkably, a q -analogue of the Taylor expansion has been introduced in Ref. [4] by means of a *basic*-binomial (11) as

$$f(x) = f(a) + \frac{(x-a)^{(1)}}{[1]!} \mathcal{D}_x^{(q)} f(x) \Big|_{x=a} + \frac{(x-a)^{(2)}}{[2]!} \mathcal{D}_x^{(q)^2} f(x) \Big|_{x=a} + \dots , \quad (13)$$

where $\mathcal{D}_x^{(q)^2} \equiv \mathcal{D}_x^{(q)} \mathcal{D}_x^{(q)}$ and so on.

Consistently with the q -calculus, we also introduce the *basic*-integration

$$\int_0^{\lambda_0} f(x) d_q x = \sum_{n=0}^{\infty} \Delta_q \lambda_n f(\lambda_n) , \quad (14)$$

where $\Delta_q \lambda_n = \lambda_n - \lambda_{n+1}$ and $\lambda_n = \lambda_0 q^n$ for $0 < q < 1$ whilst $\Delta_q \lambda_n = \lambda_{n-1} - \lambda_n$ and $\lambda_n = \lambda_0 q^{-n-1}$ for $q > 1$ [5, 6, 15, 16]. Clearly, Eq.(14) is reminiscent of the Riemann quadrature formula performed now in a q -nonuniform hierarchical lattice with a variable step $\Delta_q \lambda_n$. It is trivial to verify that

$$\mathcal{D}_x^{(q)} \int_0^x f(y) d_q y = f(x) , \quad (15)$$

for any $q > 0$.

Let us now introduce the following q -deformed exponential function defined by the series

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = 1 + x + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \cdots, \quad (16)$$

which will play an important role in the framework we are introducing. The function (16) defines the *basic*-exponential, well known in the literature since a long time ago, originally introduced in the study of basic hypergeometric series [5, 6]. In this context, let us observe that definition (16) is fully consistent with its Taylor expansion, as given by Eq.(13).

The *basic*-exponential is a monotonically increasing function, $dE_q(x)/dx > 0$, convex, $d^2E_q(x)/dx^2 > 0$, with $E_q(0) = 1$ and reducing to the ordinary exponential in the $q \rightarrow 1$ limit: $E_1(x) \equiv \exp(x)$. An important property satisfied by the q -exponential can be written formally as [6]

$$E_q(x+y) = E_q(x) E_{q^{-1}}(y), \quad (17)$$

where the left hand side of Eq.(17) must be considered by means of its series expansion in terms of *basic*-binomials:

$$E_q(x+y) = \sum_{k=0}^{\infty} \frac{(x+y)^{(k)}}{[k]!}. \quad (18)$$

By observing that $(x-x)^{(k)} = 0$ for any $k > 0$, since $(x-x)^{(0)} = 1$, from Eq.(17) we can see that [15]

$$E_q(x) E_{q^{-1}}(-x) = 1. \quad (19)$$

The above property will be crucial in the following introduction to a consistent q -deformed quantum mechanics.

Among many properties, it is important to recall the following relation [6]

$$\mathcal{D}_x^{(q)} E_q(ax) = a E_q(ax), \quad (20)$$

and its dual

$$\int_0^x E_q(ay) d_q y = \frac{1}{a} [E_q(ax) - 1]. \quad (21)$$

Finally, it should be pointed out that Eqs.(20) and (21) are two important properties of the *basic*-exponential which turns out to be not true if we employ the ordinary derivative or integral.

3. Classical q -deformed kinetic equation

Starting from the realization of the q -algebra, defined in Eq.s(2)-(3), we can write for homogeneous system the following q -deformed Fokker-Planck equation [31]

$$\frac{\partial f_q(x,t)}{\partial t} = \mathcal{D}_x^{(q)} \left[-J_1^{(q)}(x) + J_2^{(q)} \mathcal{D}_x^{(q)} \right] f_q(x,t), \quad (22)$$

where $J_1^{(q)}(x)$ and $J_2^{(q)}$ are the drift and diffusion coefficients, respectively.

The above equation has stationary solution $f_{\text{st}}^{(q)}(x)$ that can be written as

$$f_{\text{st}}^{(q)}(x) = N_q E_q[-\Phi_q(x)] , \quad (23)$$

where N_q is a normalization constant, $E_q[x]$ is the q -deformed exponential function defined in Eq.(16) and we have defined \ddagger

$$\Phi_q(x) = -\frac{1}{J_2^{(q)}} \int_0^x J_1^{(q)}(y) d_q y . \quad (24)$$

If we postulate a generalized Brownian motion in a q -deformed classical dynamics by mean the following definition of the drift and diffusion coefficients

$$J_1^{(q)}(x) = -\gamma x (q D_x^{(q)} + 1) , \quad J_2^{(q)} = \gamma/\alpha , \quad (25)$$

where γ is the friction constant, α is a constant depending on the system and $D_x^{(q)}$ is the dilatation operator (5), the stationary solution $f_{\text{st}}^{(q)}(x)$ of the above Fokker-Planck equation can be obtained as solution of the following stationary q -differential equation

$$\mathcal{D}_x^{(q)} f_{\text{st}}^{(q)}(x) = -\alpha x [q f_{\text{st}}^{(q)}(qx) + f_{\text{st}}^{(q)}(x)] . \quad (26)$$

It easy to show that the solution of the above equation can be written as

$$f_{\text{st}}^{(q)}(x) = N_q E_q[-\alpha x^2] . \quad (27)$$

4. q -deformed Schrödinger equation

We are now able to derive a q -deformed Schrödinger equation by means of a stochastic quantization method [32].

Starting from the following transformation of the probability density

$$f_q(x, t) = E_q \left[-\frac{\Phi_q(x)}{2} \right] \psi_q(x, t) , \quad (28)$$

where $\Phi_q(x)$ is the function defined in Eq.(24), the q -deformed Fokker-Planck equation (22) can be written as

$$\frac{\partial \psi_q(x, t)}{\partial t} = J_2^{(q)} \mathcal{D}_x^{(q)2} \psi_q(x, t) - V_q(x) \psi_q(x, t) , \quad (29)$$

where

$$V_q(x) = \left\{ \frac{1}{2} \mathcal{D}_x^{(q)} J_1^{(q)}(x) + \frac{[J_1^{(q)}(x)]^2}{4 J_2^{(q)}} \right\} . \quad (30)$$

The above equation has the same structure of the time dependent Schrödinger equation. In fact, the stochastic quantization of the Eq.(22) can be realized with the transformations

$$t \rightarrow \frac{t}{-i\hbar} , \quad (31)$$

$$J_2^{(q)} \rightarrow \frac{\hbar^2}{2m} , \quad (32)$$

\ddagger In the following, for simplicity, we limit ourselves to consider the drift coefficient as a monomial function of x .

getting the q -generalized Schrödinger equation

$$i\hbar \frac{\partial \psi_q(x, t)}{\partial t} = H_q \psi_q(x, t), \quad (33)$$

where

$$H_q = -\frac{\hbar^2}{2m} \mathcal{D}_x^{(q)^2} + V_q(x), \quad (34)$$

is the q -deformed Hamiltonian. Let us note that the Hamiltonian (34) is a not-Hermitian operator with respect to the standard definition based on the ordinary (undeformed) scalar product of square-integrable functions [9, 14]. In the following section, we will see as this aspect can be overridden by means the introduction of a q -deformed scalar product and generalized properties of operators inspired to q -calculus.

The above equation admits factorized solution $\psi_q(x, t) = \phi(t) \varphi_q(x)$, where $\phi(t)$ satisfies to the equation

$$i\hbar \frac{d\phi(t)}{dt} = E\phi(t), \quad (35)$$

with the standard (undeformed) solution

$$\phi(t) = \exp\left(-\frac{i}{\hbar} E t\right), \quad (36)$$

while $\varphi_q(x)$ is the solution of time-independent q -Schrödinger equation

$$H_q \varphi_q(x) = E \varphi_q(x). \quad (37)$$

In one dimensional case, for a free particle ($V_q = 0$) described by the wave function $\varphi_q^f(x)$, Eq.(37) becomes

$$\mathcal{D}_x^{(q)^2} \varphi_q^f(x) + k^2 \varphi_q^f(x) = 0, \quad (38)$$

where $k = \sqrt{2mE/\hbar^2}$. The solution of the previous equation can be written as

$$\varphi_q^f(x) = N E_q(ikx). \quad (39)$$

The above equation generalizes the plane wave function in the framework of the q -calculus.

5. q -deformed products and q -Hermitian operators

In order to develop a consistent deformed quantum dynamics, we have to generalize the products between functions and properties of the operators in the framework of the q -calculus.

Let us start on the basis of Eq.(19), which implies

$$E_q(ix) (E_{q^{-1}}(ix))^* = 1, \quad (40)$$

$$E_q(ix)^* = (E_{q^{-1}}(ix))^{-1}, \quad (41)$$

and in terms of the q -plane wave (39)

$$\varphi_{q^{-1}}^f(x)^* \varphi_q^f(x) = N^2. \quad (42)$$

Inspired to the above equation, it appears natural to introduce the complex q -conjugation of a function as

$$\psi_q^\dagger(x) = \psi_{q^{-1}}^*(x), \quad (43)$$

and, consequently, the probability density of a single particle in a finite space as

$$\rho_q(x, t) = |\psi_q(x, t)|_q^2 = \psi_q^\dagger(x, t) \psi_q(x, t) \equiv \psi_{q^{-1}}^*(x, t) \psi_q(x, t). \quad (44)$$

Thus, the wave functions must be q -square-integrable functions of configuration space, that is to say the functions $\psi_q(x)$ such that the integral

$$\int |\psi_q(x)|_q^2 d_q x, \quad (45)$$

converges.

The function space define above it is a linear space. If ψ_q and φ_q are two q -square-integrable functions, any linear combinations $\alpha\psi_q + \beta\varphi_q$, where α and β are arbitrarily chosen complex numbers, are also q -square-integrable functions.

Following this line, it is possible to define a q -scalar product of the function ψ by the function φ as

$$\langle \varphi, \psi \rangle_q = \int \varphi_q^\dagger(x) \psi_q(x) d_q x \equiv \int \varphi_{q^{-1}}^*(x) \psi_q(x) d_q x. \quad (46)$$

This is linear with respect to ψ , the norm of a function ψ_q is a real, non-negative number: $\langle \psi, \psi \rangle_q \geq 0$ and

$$\langle \psi, \varphi \rangle_q = \langle \varphi, \psi \rangle_q^\dagger. \quad (47)$$

Analogously to the undeformed case, it is easy to see that from the above properties of the q -scalar product follows the q -Schwarz inequality

$$|\langle \varphi, \psi \rangle_q|_q^2 \leq \langle \varphi, \varphi \rangle_q \langle \psi, \psi \rangle_q. \quad (48)$$

Consistently with the above definitions, the q -adjoint of an operator A_q is defined by means of the relation

$$\langle \psi, A_q^\dagger \varphi \rangle_q = \langle \varphi, A_q \psi \rangle_q^\dagger, \quad (49)$$

and, by definition, a linear operator is q -Hermitian if it is its own q -adjoint. More explicitly, an operator A_q is q -Hermitian if for any two states φ_q and ψ_q we have

$$\langle \varphi, A_q \psi \rangle_q = \langle A_q \varphi, \psi \rangle_q. \quad (50)$$

First of all, the above properties are crucial to have a consistent conservation in time of the probability densities, defined in Eq.(44). In fact, by taking the complex q -conjugation of Eq.(33), summing and integrating term by term the two equations, we get

$$i\hbar \frac{\partial}{\partial t} \int \psi_q^\dagger \psi_q d_q x = \int [\psi_{q^{-1}}^*(H_q \psi_q) - (H_{q^{-1}} \psi_{q^{-1}}^*) \psi_q] d_q x = 0, \quad (51)$$

where the last equivalence follows from the fact that the operator Hamiltonian is q -Hermitian. In this context, it is relevant to observe that it is possible to verify the

above property by using the time-spatial factorization solution $\psi_q(x, t) = \phi(t) \varphi_q(x)$ of the q -Schrödinger equation. In fact, we have

$$i\hbar \frac{\partial}{\partial t} \int \psi_q^\dagger \psi_q d_q x = \int \phi^* \phi [\varphi_{q^{-1}}^* (H_q \varphi_q) - (H_{q^{-1}} \varphi_{q^{-1}}^*) \varphi_q] d_q x. \quad (52)$$

From the stationary Schrödinger equation (37) and its complex q -conjugation we have directly

$$\varphi_{q^{-1}}^* (H_q \varphi_q) = (H_{q^{-1}} \varphi_{q^{-1}}^*) \varphi_q, \quad (53)$$

and the terms in the square bracket of Eq.(52) goes to zero.

6. Observables in q -deformed quantum mechanics

On the basis of the above properties, we have the recipe to generalize the definition of observables in the framework of q -deformed theory by postulating that:

- with the dynamical variable $A(x, p)$ is associate the linear operator $A_q(x, -i\hbar \mathcal{D}_x^{(q)})$;
- the mean value of this dynamical variable, when the system is in the dynamical (normalized) state ψ_q , is

$$\langle A \rangle_q = \int \psi_q^\dagger A_q \psi_q d_q x \equiv \int \psi_{q^{-1}}^* A_q \psi_q d_q x. \quad (54)$$

Observables are real quantities, hence the expectation value (54) must be real for any state ψ_q :

$$\int \psi_q^\dagger A_q \psi_q d_q x = \int (A_q \psi_q)^\dagger \psi_q d_q x, \quad (55)$$

therefore, on the basis of Eq.(50), observables must be represented by q -Hermitian operators.

If we require there is a state ψ_q for which the result of measuring the observable A is unique, in other words that the fluctuations

$$(\Delta A_q)^2 = \int \psi_q^\dagger (A_q - \langle A \rangle_q)^2 \psi_q d_q x, \quad (56)$$

must vanish, we obtain the following q -eigenvalue equation of a q -Hermitian operator A_q with eigenvalue a

$$A_q \varphi_q = a \varphi_q. \quad (57)$$

As a consequence, the eigenvalues of a q -Hermitian operator are real because $\langle A \rangle_q$ is real for any state; in particular for an eigenstate with the eigenvalue a for which $\langle A \rangle_q = a$.

Furthermore, as in the undeformed case, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ of the q -Hermitian operator A_q , corresponding to different eigenvalues a_1 and a_2 , are orthogonal. We can always normalize the eigenfunction, therefore we can chose all the eigenvalues of a q -Hermitian operator orthonormal, i.e.

$$\int \psi_{q,n}^\dagger \psi_{q,m} d_q x = \delta_{n,m}. \quad (58)$$

Consequently, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ belonging to different eigenvalues are linearly independent.

It is easy to see that, adapting step by step the undeformed case to the introduced q -deformed framework, the totality of the linearly independent eigenfunctions $\{\psi_{q,n}\}$ of q -Hermitian operator A_q form a complete (orthonormal) set in the space of the previously introduced q -square-integrable functions. In other words, if ψ_q is any state of a system, then it can be expanded in terms of the eigenfunctions (with a discrete spectrum) of the corresponding q -Hermitian operator A_q associate to the observable:

$$\psi_q = \sum_n c_{q,n} \psi_{q,n}, \quad (59)$$

where

$$c_{q,n} = \int \psi_{q,n}^\dagger \psi_q d_q x. \quad (60)$$

The above expansion allows us, as usual, to write the expectation value of A_q in the normed state ψ_q as

$$\langle A \rangle_q = \int \psi_q^\dagger A_q \psi_q d_q x = \sum_n |c_{q,n}|_q^2 a_n, \quad (61)$$

where $\{a_n\}$ are the set of eigenvalues (assumed, for simplicity, discrete and non-degenerate) and the normalization condition of the wave function can be written in the form

$$\sum_n |c_{q,n}|_q^2 = 1. \quad (62)$$

7. Conclusions

On the basis of the stochastic quantization procedure and on the q -differential calculus, we have obtained a generalized linear Schrödinger equation which involves a q -deformed Hamiltonian that is non-Hermitian with respect to the standard (undeformed) definition. However, under an appropriate generalization of the operators properties and the introduction of a q -deformed scalar product in the space of q -square-integral wave functions, such equation of motion satisfies the basic quantum mechanics assumptions.

Although a complete physical and mathematical description of the introduced quantum dynamical equations lies out the scope of this paper, we think that the results derived here appear to provide a deeper insight into a full consistent q -deformed quantum mechanics in the framework of the q -calculus and may be a relevant starting point for future investigations.

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References

- [1] Baxter R 1982 *Exact Solved Models in Statistical Mechanics*, (New York, Accademic Press)
- [2] Biedenharn L 1989 J. Phys. A: Math. Gen. **22** L873
- [3] Macfarlane A 1989 J. Phys. A: Math. Gen. **22** 4581
- [4] Jackson F H 1909 Am. J. Math. **38** 26; 1909 Mess. Math. **38** 57
- [5] Gasper G and Rahman M 1990 *Basic hypergeometric series*, Encyclopedia of mathematics and its applications (Cambridge Univeristy Press)
- [6] Exton H 1983 *q-Hypergeometric functions and applications* (Chichester: Ellis Horwood)
- [7] Wess J and Zumino B 1990 Nucl. Phys. B (PS) **18** 302
- [8] Celeghini E *et al.* 1995 Ann. Phys. **241** 50
- [9] Cerchiai B L, Hinterding R, Madore J and Wess J 1999 Eur. Phys. J. C **8** 547; 1999 Eur. Phys. J. C **8** 533
- [10] Bardek V, Meljanac S 2000 Eur. Phys. J. C **17** 539
- [11] Finkelstein R 1996 J. Math. Phys. **37** 983; 1996 J. Math. Phys. **37** 2628
- [12] Lavagno A and Swamy N P 2000 Phys. Rev. E **61** 1218; 2002 Phys. Rev. E **65** 036101
- [13] Lavagno A 2002 Phys. Lett. A **301** 13; Physica A **305** 238
- [14] Lavagno A, Scarfone A M and Swamy N P 2006 Eur. Phys. J. C **47** 253
- [15] Lavagno A, Scarfone A M and Swamy N P 2007 J. Phys. A: Math. Theor. **40** 8635
- [16] Erzan A and Eckmann J -P 1997 Phys. Rev. Lett. **78** 3245
- [17] Ginsburg V and Pitayevski L 1958 Sov. Phys. JETP **7** 858
- [18] Gross E P 1963 J. Math. Phys **4** 195 195
- [19] Doebner H D and Goldin G 1992 Phys. Lett. A **162** 397
- [20] Sutherland B 1985 *Lectures Notes in Physics* Springer-Verlag, Berlin Vol. 242
- [21] Mostafazadeh A 2002 J. Math. Phys. 2002 **43** 3944
- [22] Bender C M, Brody D C, Jones H F 2002 Phys. Rev. Lett. **89** 270401
- [23] Mostafazadeh A 2003 J. Phys. A:Math. Gen. **33** 7081
- [24] Bender C M, Brody D C, Jones H F 2004 Phys. Rev. Lett. **93** 251601
- [25] Jones H F 2005 J. Phys. A:Math. Gen. **38** 1741
- [26] Jones H F and Mateo J 2006 Phys. Rev. D **73** 085002
- [27] Bender C M, Brody D C, Jones H F, Meister B K 2007 Phys. Rev. Lett. **98** 040403
- [28] Zhang J -zu 1998 Phys. Lett. B **440** 66; 2000 Phys. Lett. B **477** 361
- [29] Micu M 1999 J. Phys. A **32** 7765
- [30] Ubriaco M R 1992 J. Phys. A: Math. Gen. **25** 169
- [31] Lavagno A, Scarfone A M and Swamy N P 2006 Eur. Phys. J. B **50** 351
- [32] Risken H 1989 *The Fokker-Planck Equation*, (Berlin, Springer-Verlag)